classes. One of them may be the conjugate of C, but each of the others, if $p > 2$, must contain one of the elements of C by IV, and that must be one of the C's and not A_1 or B_1. On the other hand, no two of them can contain the same C for they already have B_x in common. Therefore, there must be at least $p - 2$ of the C's, and, as k is not greater than $p - 2$, it follows that $k = p - 2$ and that C contains the same number of elements as A.

It follows also that one of these classes is the conjugate of C; that is, that the conjugate of C contains B_x, which is any one of the B's except B_1. This means that there can be only one B besides B_1, and that the class B can contain only two elements.

Now C was any class non-conjugate to A. Therefore, every class except B contains p elements while B contains only two. But if we start with some other class in place of A, we can prove that its conjugate contains two elements and that B contains p. This is possible only if $p = 2$ and every class contains only two elements.

THEOREM 5. *The elements of S and the m-classes are isomorphic with the elements A, B, C, D and the sets AB, AC, AD, BC, BD, CD.*

The four elements of theorem 2 may be labelled A, B, C, D and the six m-classes of theorem 3 the six of this proposition. By theorem 4 there are no more elements in any of these m-classes, and by II no more m-classes containing only these elements. Let X be an additional element. By I there is a class AX, and by theorem 4, it contains no more elements. We then have classes AX and AB both conjugate to CD, and since this violates IV it proves that there are no elements X. The six sets above evidently satisfy I to VII.

NOTE. In connection with the above solution Professor Veblen's comments on the origin and discussion of the problem will be of interest to the readers of the MONTHLY.

Professor VEBLEN says, "The problem originated in my course in Projective Geometry which I began, as usual, by a discussion of the abstract point of view in mathematics. In order to emphasize the point that our logical processes should be independent of any particular set of mental images or, indeed, of any knowledge of what the propositions are about, I proposed that certain members of the class should make up a set of postulates which would give the properties of some set of objects chosen but not divulged by them. The other students were then challenged to make logical deductions from the postulates and thereby deduce enough theorems to learn what the postulate makers had in mind.

"As a result of this suggestion, Mr. Post and Mr. Franklin brought forward a set of postulates which is essentially the one offered in the enclosed problem and a solution was found by several of the other students in precisely the manner indicated. One of the students indeed went further and pointed out that one of the postulates in the original set was redundant.

"The exercise was a success in showing how mathematical deductions can be made without knowledge as to what one is reasoning about. It also brought out vividly the problem of the significance of the logical processes as a method of discovery. While there is a sense in which it is true that you cannot get anything out of a set of postulates except what has been put in them, you can at least find out what was put in them. This is what the solver of this problem has to do in a simple case. In the more complicated case of ordinary geometry the student is apt to think that he understands what is in the axioms, but every time that he witnesses the derivation of a new theorem it turns out that there was something in them that he had not seen before.

"By a slight modification this problem can also be used to propose another problem which it seems to me may turn out to be an important one. Let us replace Assumption VII by the assumption that no m-class contains more than 6 elements. The mathematical science based on the assumptions could then be built up without ever counting beyond, let us say, 24. The question arises, how much of logic is needed to develop so limited a mathematical science? Would it be possible to single out a subset of the postulates of logic which would suffice for the purpose? If so, what processes of logic can be omitted? What sort of a logic results if the omitted processes are replaced by others? If it should turn out that the logic required for a satisfying theory of this finite system (or any other partciular system as, for example, a particular finite projective space) stops short of that required for larger systems, we would be in the presence of a criterion for the classification of logical processes which might help toward deciding the question as to what logical processes are legitimate in dealing with various types of infinite sets."

2899 [1921, 228]. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+ 60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.

I. Solution by C. F. Gummer, Queen's University.

These are particular cases of a more general theorem. Consider n coplanar points $A_1, A_2, \cdots, A_n$; and let a point in the plane, starting at P, revolve successively about $A_1, A_2, \cdots, A_n, A_1, \cdots$ through angles $\phi_1, \phi_2, \cdots, \phi_n, \phi_1, \cdots$, until all the points $A_1, \cdots, A_n$ have been used m times each. The resultant displacement from P after the first set of rotations about $A_1, A_2, \cdots, A_n$ is the sum of a number of vectors, namely PA_1, A_1A_n and those obtained by turning A_nA_{n-1} through ϕ_n, $A_{n-1}A_{n-2}$ through $\phi_{n-1} + \phi_n$, $\cdots$ A_2A_1 through $\phi_2 + \phi_3 + \cdots \phi_n$ and finally A_1P through $\phi_1 + \phi_2 + \cdots \phi_n$. If Φ denotes the sum of the n angles, the vector sum after m sets of rotations contains two vectors of length PA_1 inclined at an angle $\pi + m\Phi$, and it contains m vectors of length A_iA_{i-1} (including the case of A_1A_n) inclined successively at the angle Φ so that they may be regarded as a series of equal chords placed end to end in a circle of suitable size. This is true for each value of i from 2 to n and for A_1A_n.

Three cases occur:

(1) *$\Phi/(2\pi)$ is rational but not integral.* A value of m other than 1 may then be found so that $m\Phi$ is a multiple of 2π. It follows that after m sets of rotations the two vectors derived from PA_1 are equal and opposite, and the vectors derived from A_iA_{i-1} are the sides of a closed regular polygon (possibly interlacing); therefore the resultant displacement vanishes, and the point has moved in a closed curve. The problems proposed belong to this case.

(2) *$\Phi/(2\pi)$ is irrational.* No group of vectors can be made to have a zero sum; and it will be found that the path cannot be closed (except for special positions of P). The path however lies in a finite region, since each group of vectors has a sum not greater than the diameter of its corresponding circle; and it may be shown that the path returns to positions indefinitely near to the initial point.

(3) *$\Phi/(2\pi)$ is an integer.* On taking $m = 1$, A_1P and PA_1 again destroy one another; but the remaining vectors, being now in groups of one, have not generally a zero sum. The resultant transformation is a translation of the entire plane, which by repetition (unless it happens to vanish) carries every point to infinity.

II. Solution by A. A. Bennett, University of Texas.

The problem will be treated in the more general case as follows: Given m and n two positive integers each greater than unity. Let $A_1, A_2, \cdots A_m, B_1$ be any $m + 1$ coplanar points. Let P describe a curve starting from B_1, made up of circular arcs, each of which is a one-(mn)th part of a circumference but with various radii as follows: With center A_1 describe one-(mn)th part of a circumference, positively from B_1, terminating at B_2. With center, A_2, [describe one-(mn)th part of a circumference positively from B_2, terminating at B_3. Continue cyclically taking as the $(m + 1)$st center A_1, $(m + 2)$nd center A_2, etc. Show that B_{mn+1} coincides with B_1.

Let us use vector methods and denote the positive turn of one-(mn)th part of a circumference by the operator T. Then $T^{mn} = 1$. We shall then have the following relations:[1]

$$\begin{aligned} B_2 - A_1 &= T(B_1 - A_1), \\ B_3 - A_2 &= T(B_2 - A_2), \\ &\cdots\cdots\cdots \\ B_{k+1} - A_k &= T(B_k - A_k), \\ &\cdots\cdots\cdots \end{aligned}$$

Collecting terms,

$$B_{m+1} = T^m B_1 + (1 - T)(T^{m-1}A_1 + T^{m-2}A_2 + \cdots + TA_{m-1} + A_m)$$

and finally,

$$B_{mn+1} = T^{mn}B_1 + (1 + T^m + T^{2m} + \cdots + T^{(n-1)m})(1 - T)(T^{m-1}A_1 + T^{m-2}A_2 + \cdots + TA_{m-1} + A_m).$$

Since $T^m \neq 1$, and $(T^m - 1)(1 + T^m + T^{2m} + \cdots + T^{(n-1)m}) = T^{mn} - 1 = 0$, it follows that

[1] In this notation we may regard the difference of two points as a vector and the sum of a point and a vector as a point after the manner of Grassmann (see E. W. Hyde, *The Directional Calculus*, Boston, 1890, p. 2), but when we apply the distributive law to T, writing $T(B - A)$, for example, as $TB - TA$, we must understand A and B to represent vectors drawn from some arbitrary point O; that is, we may say that TB and TA stand for $T(B - O)$ and $T(A - O)$.

the operator $(1 + T^m + T^{2m} + \cdots + T^{(n-1)m})$ is a null-operator, so that

$$B_{mn+1} = T^{mn}B_1 = B_1$$

as desired.

It is of interest to carry out the work for an infinite number of points.

Let $x(t)$, $y(t)$ be the coördinates of a closed curve as the parameter t ranges from zero to 2π and be periodic with period 2π. Let $X(t)$ and $Y(t)$ be the coördinates of a derived curve obtained by a limiting process from the above discussion. Let us describe an infinitesimal circular arc with $x + \Delta t\cdot x'$, and $y + \Delta t\cdot y'$ as coördinates of the center, starting from the point with coördinates $X(t)$ and $Y(t)$, and described positively with an angle equal to the circumference divided by $n/(\Delta t/2\pi)$. The terminal point of the arc will be denoted by $(X + \Delta t\cdot X', Y + \Delta t\cdot Y')$. Since the arc is circular, we shall have

$$[(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')]^2 + [(X + \Delta t\cdot X') - (x + \Delta t\cdot x')]^2 = [Y - (y + \Delta t\cdot y')]^2 + [X - (x + \Delta t\cdot x')]^2. \tag{1}$$

The slope of the initial position of the radius is $[Y - (y + \Delta t\cdot y')]/[X - (x + \Delta t\cdot x')]$, and of the terminal position is $[(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')]/[(X + \Delta t\cdot X') - (x + \Delta t\cdot x')]$. The tangent of the angle of rotation is to be equal to $\tan(\Delta t/n)$; thus,

$$\frac{\dfrac{(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')}{(X + \Delta t\cdot X') - (x + \Delta t\cdot x')} - \dfrac{Y - (y + \Delta t\cdot y')}{X - (x + \Delta t\cdot x')}}{1 + \dfrac{(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')}{(X + \Delta t\cdot X') - (x + \Delta t\cdot x')}\cdot\dfrac{Y - (y + \Delta t\cdot y')}{X - (x + \Delta t\cdot x')}} = \tan\left(\frac{\Delta t}{n}\right). \tag{2}$$

Simplifying and dropping higher powers of Δt, we have from (2)

$$\frac{\dfrac{Y-y}{X-x}\left[1 + \Delta t\left(\dfrac{Y'-y'}{Y-y} - \dfrac{X'-x'}{X-x}\right)\right] - \dfrac{Y-y}{X-x}\left[1 - \Delta t\left(\dfrac{y'}{Y-y} - \dfrac{x'}{X-x}\right)\right]}{1 + \left(\dfrac{Y-y}{X-x}\right)^2} = \frac{\Delta t}{n}$$

or

$$Y'(X - x) - X'(Y - y) = [(Y - y)^2 + (X - x)^2]/n. \tag{3}$$

From (1), we have similarly,

$$Y'(Y - y) + X'(X - x) = 0; \tag{4}$$

whence,

$$-nX' = Y - y, \qquad nY' = X - x. \tag{5}$$

Eliminating between these we have the following pair of equations, to determine $X(t)$ and $Y(t)$, in terms of $x(t)$ and $y(t)$:

$$n^2X'' + X = x + ny', \qquad n^2Y'' + Y = y - nx'. \tag{6}$$

These are to be taken subject to the initial condition that when $t = 0$, $X(0)$ and $Y(0)$ have assigned values, and, from (5)

$$-nX'(0) = Y(0) - y(0), \qquad ny'(0) = X(0) - x(0). \tag{7}$$

The solutions are therefore determined.

Since $x(t)$ and $y(t)$ are periodic of period 2π, and since the solution of the homogeneous equations, $n^2X'' + X = 0$, $n^2Y'' + Y = 0$, are periodic of period $2n\pi$, it follows for this example that X and Y have each the period $2n\pi$. Thus for this "infinite" case also the set of derived points closes after the original set is described cyclically n times. It is to be noted that for each new choice of a variable t, the given curve whose parametric equations are $x = x(t)$, $y = y(t)$, is regarded as the limit of a new finite set of points.

NOTE BY THE EDITORS.—Professor Gummer's solution differs from the first part of Professor Bennett's only in that the rotations are different, making his solution more general.

These solutions may also be expressed in terms of complex quantities without any introduction of the notion of vectors or any operator T.

Also solved by T. M. BLAKSLEE and F. L. WILMER. Professor Blakslee sent in four different solutions, one of which was the generalized solution.